Tutorial on Convex Optimization for Engineers
Part II

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Outline

1. Introduction

2. Convex sets

3. Convex functions

4. Convex optimization problems

5. Lagrangian duality theory

6. Disciplined convex programming and CVX
4. Convex optimization problems
Optimization problem in standard form

minimize \( f_0(x) \)
subject to \( f_i(x) \leq 0, \quad i = 1, \ldots, m \)
\quad \quad \quad \quad h_i(x) = 0, \quad i = 1, \ldots, p \)

- \( x \in \mathbb{R}^n \) is the optimization variable
- \( f_0 : \mathbb{R}^n \to \mathbb{R} \) is the objective or cost function
- \( f_i : \mathbb{R}^n \to \mathbb{R}, \ i = 1, \ldots, m, \) are the inequality constraint functions
- \( h_i : \mathbb{R}^n \to \mathbb{R} \) are the equality constraint functions

optimal value:

\[ p^* = \inf\{f_0(x) \mid f_i(x) \leq 0, \ i = 1, \ldots, m, \ h_i(x) = 0, \ i = 1, \ldots, p\} \]

- \( p^* = \infty \) if problem is infeasible (no \( x \) satisfies the constraints)
- \( p^* = -\infty \) if problem is unbounded below
Feasibility problem

find \( x \)
subject to \( f_i(x) \leq 0, \quad i = 1, \ldots, m \)
\( h_i(x) = 0, \quad i = 1, \ldots, p \)

can be considered a special case of the general problem with \( f_0(x) = 0 \):

minimize \( 0 \)
subject to \( f_i(x) \leq 0, \quad i = 1, \ldots, m \)
\( h_i(x) = 0, \quad i = 1, \ldots, p \)

- \( p^* = 0 \) if constraints are feasible; any feasible \( x \) is optimal
- \( p^* = \infty \) if constraints are infeasible
Convex optimization problem

standard form convex optimization problem

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad a_i^T x = b_i, \quad i = 1, \ldots, p
\end{align*}
\]

- \( f_0, f_1, \ldots, f_m \) are convex; equality constraints are affine
- problem is quasiconvex if \( f_0 \) is quasiconvex (and \( f_1, \ldots, f_m \) convex)

often written as

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

important property: feasible set of a convex optimization problem is convex
Equivalent convex problems

two problems are (informally) equivalent if the solution of one is readily obtained from the solution of the other, and vice-versa

some common transformations that preserve convexity:

• eliminating equality constraints

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\text{minimize (over } z) & \quad f_0(Fz + x_0) \\
\text{subject to} & \quad f_i(Fz + x_0) \leq 0, \quad i = 1, \ldots, m
\end{align*}
\]

where \( F \) and \( x_0 \) are such that

\[
Ax = b \iff x = Fz + x_0 \text{ for some } z
\]
• introducing equality constraints

\[
\begin{align*}
\text{minimize} & \quad f_0(A_0x + b_0) \\
\text{subject to} & \quad f_i(A_ix + b_i) \leq 0, \quad i = 1, \ldots, m
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\text{minimize (over } x, y_i) & \quad f_0(y_0) \\
\text{subject to} & \quad f_i(y_i) \leq 0, \quad i = 1, \ldots, m \\
y_i = A_ix + b_i, & \quad i = 0, 1, \ldots, m
\end{align*}
\]

• introducing slack variables for linear inequalities

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad a_i^Tx \leq b_i, \quad i = 1, \ldots, m
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\text{minimize (over } x, s) & \quad f_0(x) \\
\text{subject to} & \quad a_i^Tx + s_i = b_i, \quad i = 1, \ldots, m \\
s_i \geq 0, & \quad i = 1, \ldots, m
\end{align*}
\]
- **epigraph form**: standard form convex problem is equivalent to

\[
\begin{align*}
\text{minimize } & (\text{over } x, t) & & t \\
\text{subject to } & f_0(x) - t & \leq & 0 \\
& f_i(x) & \leq & 0, \quad i = 1, \ldots, m \\
& Ax & = & b
\end{align*}
\]

- **minimizing over some variables**

\[
\begin{align*}
\text{minimize } & f_0(x_1, x_2) \\
\text{subject to } & f_i(x_1) & \leq & 0, \quad i = 1, \ldots, m
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\text{minimize } & \tilde{f}_0(x_1) \\
\text{subject to } & f_i(x_1) & \leq & 0, \quad i = 1, \ldots, m
\end{align*}
\]

where \( \tilde{f}_0(x_1) = \inf_{x_2} f_0(x_1, x_2) \)
Linear program (LP)

\[
\begin{align*}
\text{minimize} & \quad c^T x + d \\ 
\text{subject to} & \quad Gx \preceq h \\ & \quad Ax = b
\end{align*}
\]

- convex problem with affine objective and constraint functions
- feasible set is a polyhedron
Examples

diet problem: choose quantities $x_1, \ldots, x_n$ of $n$ foods

- one unit of food $j$ costs $c_j$, contains amount $a_{ij}$ of nutrient $i$
- healthy diet requires nutrient $i$ in quantity at least $b_i$

to find cheapest healthy diet,

$$\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \succeq b, \quad x \succeq 0
\end{align*}$$

piecewise-linear minimization

$$\begin{align*}
\text{minimize} & \quad \max_{i=1,\ldots,m}(a_i^T x + b_i) \\
\text{equivalent to an LP} & \\
\text{minimize} & \quad t \\
\text{subject to} & \quad a_i^T x + b_i \leq t, \quad i = 1, \ldots, m
\end{align*}$$
Quadratic program (QP)

minimize \( \frac{1}{2}x^T P x + q^T x + r \)
subject to \( Gx \preceq h \)
\( Ax = b \)

- \( P \in S^n_+ \), so objective is convex quadratic
- minimize a convex quadratic function over a polyhedron

Convex optimization problems

Convex optimization problems
Examples

least-squares

minimize \[ \|Ax - b\|^2 \]

• analytical solution \( x^* = A^\dagger b \) (\( A^\dagger \) is pseudo-inverse)

• can add linear constraints, e.g., \( l \leq x \leq u \)

linear program with random cost

\[
\text{minimize } \bar{c}^T x + \gamma x^T \Sigma x = \mathbf{E} c^T x + \gamma \text{var}(c^T x)
\]

subject to \( Gx \leq h,\ Ax = b \)

• \( c \) is random vector with mean \( \bar{c} \) and covariance \( \Sigma \)

• hence, \( c^T x \) is random variable with mean \( \bar{c}^T x \) and variance \( x^T \Sigma x \)

• \( \gamma > 0 \) is risk aversion parameter; controls the trade-off between expected cost and variance (risk)
Quadratically constrained quadratic program (QCQP)

\[
\begin{align*}
\text{minimize} & \quad (1/2)x^T P_0 x + q_0^T x + r_0 \\
\text{subject to} & \quad (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \ldots, m \\
\text{subject to} & \quad Ax = b
\end{align*}
\]

- \( P_i \in S^n_+ \); objective and constraints are convex quadratic

- if \( P_1, \ldots, P_m \in S^n_{++} \), feasible region is intersection of \( m \) ellipsoids and an affine set
Second-order cone programming

minimize \( f^T x \)
subject to \( \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m \)
\( F x = g \)

\( (A_i \in \mathbb{R}^{n_i \times n}, \ F \in \mathbb{R}^{p \times n}) \)

- inequalities are called second-order cone (SOC) constraints:
  \( (A_i x + b_i, c_i^T x + d_i) \in \text{second-order cone in} \ \mathbb{R}^{n_i + 1} \)

- for \( n_i = 0 \), reduces to an LP; if \( c_i = 0 \), reduces to a QCQP
- more general than QCQP and LP
Semidefinite program (SDP)

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0 \\
& \quad Ax = b
\end{align*}
\]

with \( F_i, G \in S^k \)

- inequality constraint is called linear matrix inequality (LMI)
- includes problems with multiple LMI constraints: for example,

\[
x_1 \hat{F}_1 + \cdots + x_n \hat{F}_n + \hat{G} \preceq 0, \quad x_1 \tilde{F}_1 + \cdots + x_n \tilde{F}_n + \tilde{G} \preceq 0
\]

is equivalent to single LMI

\[
x_1 \begin{bmatrix} \hat{F}_1 & 0 \\ 0 & \hat{F}_1 \end{bmatrix} + x_2 \begin{bmatrix} \hat{F}_2 & 0 \\ 0 & \hat{F}_2 \end{bmatrix} + \cdots + x_n \begin{bmatrix} \hat{F}_n & 0 \\ 0 & \hat{F}_n \end{bmatrix} + \begin{bmatrix} \hat{G} & 0 \\ 0 & \tilde{G} \end{bmatrix} \preceq 0
\]
LP and SOCP as SDP

LP and equivalent SDP

LP: minimize $c^T x$  
subject to $Ax \preceq b$

SDP: minimize $c^T x$  
subject to $\text{diag}(Ax - b) \preceq 0$

(note different interpretation of generalized inequality $\preceq$)

SOCP and equivalent SDP

SOCP: minimize $f^T x$  
subject to $\|Ax + b\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m$

SDP: minimize $f^T x$  
subject to $\begin{bmatrix} (c_i^T x + d_i)I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{bmatrix} \succeq 0, \quad i = 1, \ldots, m$
Some nonconvex problems

slight modifications of convex problems can be very hard

- **convex maximization, concave minimization:**

  \[
  \text{maximize} \quad \|Ax - b\|^2 \\
  \text{subject to} \quad \|x\| \leq 1
  \]

- **nonlinear equality constraints:**

  \[
  \text{minimize} \quad c^T x \\
  \text{subject to} \quad x^T Q_i x + q_i^T x + c_i = 0, \quad 1 \leq i \leq K,
  \]

  where $Q_i \succeq 0$

- **minimizing over integer constraints:**

  find $x$ such that $Ax \leq b, \quad x_i \text{ is integer}$

Convex optimization problems
5. Lagrangian duality theory
**Lagrangian**

**standard form problem** (not necessarily convex)

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad h_i(x) = 0, \quad i = 1, \ldots, p
\end{align*}
\]

variable \( x \in \mathbb{R}^n \), domain \( \mathcal{D} \), optimal value \( p^* \)

**Lagrangian:** \( L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R} \), with \( \text{dom} \ L = \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^p \),

\[
L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)
\]

- weighted sum of objective and constraint functions
- \( \lambda_i \) is Lagrange multiplier associated with \( f_i(x) \leq 0 \)
- \( \nu_i \) is Lagrange multiplier associated with \( h_i(x) = 0 \)

Duality
Lagrange dual function

Lagrange dual function: \( g : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R} \),

\[
g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu)
\]

\[
= \inf_{x \in \mathcal{D}} \left( f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)
\]

\( g \) is concave, can be \(-\infty\) for some \( \lambda, \nu \)

**lower bound property:** if \( \lambda \succeq 0 \), then \( g(\lambda, \nu) \leq p^* \)

proof: if \( \tilde{x} \) is feasible and \( \lambda \succeq 0 \), then

\[
f_0(\tilde{x}) \geq L(\tilde{x}, \lambda, \nu) \geq \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = g(\lambda, \nu)
\]

minimizing over all feasible \( \tilde{x} \) gives \( p^* \geq g(\lambda, \nu) \)
Least-norm solution of linear equations

\[
\begin{align*}
\text{minimize} & \quad x^T x \\
\text{subject to} & \quad Ax = b
\end{align*}
\]

dual function

- Lagrangian is \( L(x, \nu) = x^T x + \nu^T (Ax - b) \)
- to minimize \( L \) over \( x \), set gradient equal to zero:

\[
\nabla_x L(x, \nu) = 2x + A^T \nu = 0 \implies x = -(1/2)A^T \nu
\]

- plug in in \( L \) to obtain \( g \):

\[
g(\nu) = L((-1/2)A^T \nu, \nu) = -\frac{1}{4} \nu^T A A^T \nu - b^T \nu
\]

a concave function of \( \nu \)

**lower bound property**: \( p^* \geq -(1/4)\nu^T A A^T \nu - b^T \nu \) for all \( \nu \)
The dual problem

Lagrange dual problem

\[
\begin{align*}
\text{maximize} & \quad g(\lambda, \nu) \\
\text{subject to} & \quad \lambda \succeq 0
\end{align*}
\]

- finds best lower bound on \( p^* \), obtained from Lagrange dual function
- a convex optimization problem; optimal value denoted \( d^* \)
- \( \lambda, \nu \) are dual feasible if \( \lambda \succeq 0, (\lambda, \nu) \in \text{dom } g \)
- often simplified by making implicit constraint \( (\lambda, \nu) \in \text{dom } g \) explicit

example: standard form LP and its dual

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \succeq 0 \\
\text{maximize} & \quad -b^T \nu \\
\text{subject to} & \quad A^T \nu + c \succeq 0
\end{align*}
\]
Weak and strong duality

**weak duality:** \( d^* \leq p^* \)

- always holds (for convex and nonconvex problems)
- can be used to find nontrivial lower bounds for difficult problems

  for example, solving the SDP

\[
\begin{align*}
\text{maximize} & \quad -1^T \nu \\
\text{subject to} & \quad W + \text{diag}(\nu) \succeq 0
\end{align*}
\]

  gives a lower bound for the two-way partitioning problem

**strong duality:** \( d^* = p^* \)

- does not hold in general
- (usually) holds for convex problems
- conditions that guarantee strong duality in convex problems are called **constraint qualifications**
Slater’s constraint qualification

strong duality holds for a convex problem

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

if it is strictly feasible, i.e.,

\[\exists x \in \text{int } D : \quad f_i(x) < 0, \quad i = 1, \ldots, m, \quad Ax = b\]

- also guarantees that the dual optimum is attained (if \( p^* > -\infty \))
- can be sharpened: e.g., can replace \( \text{int } D \) with \( \text{relint } D \) (interior relative to affine hull); linear inequalities do not need to hold with strict inequality, . . .
- there exist many other types of constraint qualifications
Quadratic program

primal problem (assume $P \in S^{n}_{++}$)

\[
\begin{align*}
\text{minimize} & \quad x^T P x \\
\text{subject to} & \quad Ax \preceq b
\end{align*}
\]

dual function

\[
g(\lambda) = \inf_x (x^T P x + \lambda^T (Ax - b)) = -\frac{1}{4} \lambda^T A P^{-1} A^T \lambda - b^T \lambda
\]

dual problem

\[
\begin{align*}
\text{maximize} & \quad -(1/4) \lambda^T A P^{-1} A^T \lambda - b^T \lambda \\
\text{subject to} & \quad \lambda \succeq 0
\end{align*}
\]

• from Slater’s condition: $p^* = d^*$ if $A\tilde{x} \prec b$ for some $\tilde{x}$

• in fact, $p^* = d^*$ always
Geometric interpretation

for simplicity, consider problem with one constraint $f_1(x) \leq 0$

interpretation of dual function:

$$g(\lambda) = \inf_{(u,t) \in G} (t + \lambda u), \quad \text{where} \quad G = \{(f_1(x), f_0(x)) \mid x \in D\}$$

- $\lambda u + t = g(\lambda)$ is (non-vertical) supporting hyperplane to $G$
- hyperplane intersects $t$-axis at $t = g(\lambda)$
Complementary slackness

Assume strong duality holds, $x^*$ is primal optimal, $(\lambda^*, \nu^*)$ is dual optimal.

$$f_0(x^*) = g(\lambda^*, \nu^*) = \inf_x \left( f_0(x) + \sum_{i=1}^{m} \lambda^*_i f_i(x) + \sum_{i=1}^{p} \nu^*_i h_i(x) \right)$$

$$\leq f_0(x^*) + \sum_{i=1}^{m} \lambda^*_i f_i(x^*) + \sum_{i=1}^{p} \nu^*_i h_i(x^*)$$

$$\leq f_0(x^*)$$

Hence, the two inequalities hold with equality:

- $x^*$ minimizes $L(x, \lambda^*, \nu^*)$
- $\lambda^*_i f_i(x^*) = 0$ for $i = 1, \ldots, m$ (known as complementary slackness):
  $$\lambda^*_i > 0 \implies f_i(x^*) = 0, \quad f_i(x^*) < 0 \implies \lambda^*_i = 0$$

Duality
Karush-Kuhn-Tucker (KKT) conditions

the following four conditions are called KKT conditions (for a problem with differentiable $f_i$, $h_i$):

1. primal constraints: $f_i(x) \leq 0$, $i = 1, \ldots, m$, $h_i(x) = 0$, $i = 1, \ldots, p$
2. dual constraints: $\lambda \succeq 0$
3. complementary slackness: $\lambda_i f_i(x) = 0$, $i = 1, \ldots, m$
4. gradient of Lagrangian with respect to $x$ vanishes:

$$\nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + \sum_{i=1}^{p} \nu_i \nabla h_i(x) = 0$$
KKT conditions for convex problem

if \( \tilde{x}, \tilde{\lambda}, \tilde{\nu} \) satisfy KKT for a convex problem, then they are optimal:

- from complementary slackness: \( f_0(\tilde{x}) = L(\tilde{x}, \tilde{\lambda}, \tilde{\nu}) \)
- from 4th condition (and convexity): \( g(\tilde{\lambda}, \tilde{\nu}) = L(\tilde{x}, \tilde{\lambda}, \tilde{\nu}) \)

hence, \( f_0(\tilde{x}) = g(\tilde{\lambda}, \tilde{\nu}) \)

if **Slater’s condition** is satisfied:

\( x \) is optimal if and only if there exist \( \lambda, \nu \) that satisfy KKT conditions

- recall that Slater implies strong duality, and dual optimum is attained
- generalizes optimality condition \( \nabla f_0(x) = 0 \) for unconstrained problem
6. Disciplined convex programming and CVX
Convex optimization solvers

• **LP solvers**
  - lots available (GLPK, Excel, Matlab’s `linprog`, . . .)

• **cone solvers**
  - typically handle (combinations of) LP, SOCP, SDP cones
  - several available (SDPT3, SeDuMi, CSDP, . . .)

• **general convex solvers**
  - some available (CVXOPT, MOSEK, . . .)

• plus lots of special purpose or application specific solvers

• could write your own
Transforming problems to standard form

• you’ve seen lots of tricks for transforming a problem into an equivalent one that has a standard form (e.g., LP, SDP)

• these tricks greatly extend the applicability of standard solvers

• writing code to carry out this transformation is often painful

• **modeling systems** can partly automate this step
Disciplined convex programming

- describe objective and constraints using expressions formed from
  - a set of basic atoms (convex, concave functions)
  - a restricted set of operations or rules (that preserve convexity)

- modeling system keeps track of affine, convex, concave expressions

- rules ensure that
  - expressions recognized as convex (concave) are convex (concave)
  - but, some convex (concave) expressions are not recognized as convex (concave)

- problems described using DCP are convex by construction
CVX

- uses DCP
- runs in Matlab, between the `cvx_begin` and `cvx_end` commands
- relies on SDPT3 or SeDuMi (LP/SOCP/SDP) solvers
- refer to user guide, online help for more info
- the CVX example library has more than a hundred examples
Example: Constrained norm minimization

A = randn(5, 3);
b = randn(5, 1);
cvx_begin
    variable x(3);
    minimize(norm(A*x - b, 1))
    subject to
        -0.5 <= x;
        x <= 0.3;
cvx_end

- between cvx_begin and cvx_end, x is a CVX variable
- statement subject to does nothing, but can be added for readability
- inequalities are interpreted elementwise
What CVX does

after cvx_end, CVX

- transforms problem into an LP
- calls solver SDPT3
- overwrites (object) $x$ with (numeric) optimal value
- assigns problem optimal value to cvx_optval
- assigns problem status (which here is Solved) to cvx_status

(had problem been infeasible, cvx_status would be Infeasible and $x$ would be NaN)
# Some functions

<table>
<thead>
<tr>
<th>function</th>
<th>meaning</th>
<th>attributes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{norm}(x, p)$</td>
<td>$|x|_p$</td>
<td>cvx</td>
</tr>
<tr>
<td>$\text{square}(x)$</td>
<td>$x^2$</td>
<td>cvx</td>
</tr>
<tr>
<td>$\text{square}_{\text{pos}}(x)$</td>
<td>$(x_+)^2$</td>
<td>cvx, nondecr</td>
</tr>
<tr>
<td>$\text{pos}(x)$</td>
<td>$x_+$</td>
<td>cvx, nondecr</td>
</tr>
<tr>
<td>$\text{sum}_{\text{largest}}(x,k)$</td>
<td>$x[1] + \cdots + x[k]$</td>
<td>cvx, nondecr</td>
</tr>
<tr>
<td>$\text{sqrt}(x)$</td>
<td>$\sqrt{x}$ ($x \geq 0$)</td>
<td>ccv, nondecr</td>
</tr>
<tr>
<td>$\text{inv}_{\text{pos}}(x)$</td>
<td>$1/x$ ($x &gt; 0$)</td>
<td>cvx, nonincr</td>
</tr>
<tr>
<td>$\text{max}(x)$</td>
<td>$\max{x_1, \ldots, x_n}$</td>
<td>cvx, nondecr</td>
</tr>
<tr>
<td>$\text{quad}<em>{\text{over}}</em>{\text{lin}}(x,y)$</td>
<td>$x^2/y$ ($y &gt; 0$)</td>
<td>cvx, nonincr in y</td>
</tr>
<tr>
<td>$\text{lambda}_{\text{max}}(X)$</td>
<td>$\lambda_{\text{max}}(X)$ ($X = X^T$)</td>
<td>cvx</td>
</tr>
</tbody>
</table>
| $\text{huber}(x)$   | \[
|& \begin{cases}
|& x^2, \quad |x| \leq 1 \\
|& 2|x| - 1, \quad |x| > 1
|\end{cases}
|             | cvx                                   |
References

- Course “Convex Optimization I” by Prof. Stephen Boyd at Stanford University, CA.